

Effective spatial dimension of extremal non-dilatonic black p -branes and the description of entropy on the world volume

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Abstract

By investigating the critical behavior appearing at the extremal limit of the non-dilatonic, black p -branes in $(d+p)$ dimensions, we find that some critical exponents related to the critical point obey the scaling laws. From the scaling laws we obtain that the effective spatial dimension of the non-dilatonic black holes and black strings is one, and is p for the non-dilatonic black p -branes. For the dilatonic black holes and black p -branes, the effective dimension will depend on the parameters in theories. Thus, we give an interpretation why the Bekenstein-Hawking entropy may be given a simple world volume interpretation only for the non-dilatonic black p -branes.

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Classical general relativity and the quantum field theory in curved spacetime together provide the temperature and entropy of black holes [1-3]. Although the Bekenstein-Hawking entropy of black holes can indeed be derived in the Euclidean path integral method of quantum gravity under the zero-loop approximation [4], a satisfactory statistical interpretation of the entropy is still needed. In the last few months, the important progress towards a microscopic understanding of the black hole entropy has been made. Strominger and Vafa [5] considered a class of five-dimensional extremal black holes in string theory. They found that the Bekenstein-Hawking entropy of the black holes agrees with that of BPS soliton bound states with same charges. Since then, a lot of papers appear to study this agreement in the extremal and near-extremal black holes, black strings and black p -branes (some extended objects surrounded by event horizons) [6]. Klebanov and Tseytlin [7] found that there are non-dilatonic p -branes whose near-extremal entropy may be explained by free massless fields on the world volume. Gubser, Klebanov, and Peet [6] also got the similar result in the black 3-branes. But the reason seems to be unclear.

In this Letter, by investigating the critical behavior occurring at the extremal limit ($r_- = r_+$) of the non-dilatonic, black p -branes in $(d + p)$ dimensions (the dilaton field is a constant throughout spacetime), we obtain an effective spatial dimension of the black p -branes. For the non-dilatonic black holes and black strings the effective dimension is one, and it is just p for the non-dilatonic black p -branes. When the dilatonic black holes are obtained by the double-dimensional reduction of the non-dilatonic black p -branes [8], the effective dimension is also p . For a general coupling constant a governing the interaction strength between the dilaton field and the asymmetric tensor field, the effective dimension will depend on the parameters in theories. Thus, we give an interpretation of the result of Klebanov and Tseytlin [7] that the Bekenstein-Hawking entropy may be given a simple world volume interpretation only for the non-dilatonic p -branes (including the non-dilatonic black holes).

We start with the non-dilatonic, $(d + p)$ -dimensional action [8],

$$S_{d+p} = \frac{1}{16\pi} \int d^{(d+p)}x \sqrt{-g} \left[R - \frac{2}{(d-2)!} F_{d-2}^2 \right], \quad (1)$$

where R is the scalar curvature and F_{d-2} denotes the $(d-2)$ -form asymmetric tensor field. Performing the double-dimensional reduction by p dimensions, one has the dilatonic d -dimensional action:

$$S_d = \frac{1}{16\pi} \int d^d x \sqrt{-g} \left[R - 2(\nabla\phi)^2 - \frac{2}{(d-2)!} e^{-2a\phi} F_{d-2}^2 \right], \quad (2)$$

where ϕ is the dilaton field, and the constant a is

$$a = \frac{(d-3)\sqrt{2p}}{\sqrt{(d-2)(d+p-2)}}. \quad (3)$$

The magnetically charged black holes in the action (2) are [9]

$$\begin{aligned} ds_d^2 &= \left[1 - \left(\frac{r_+}{r} \right)^{d-3} \right] \left[1 - \left(\frac{r_-}{r} \right)^{d-3} \right]^{1-(d-3)b} dt^2 \\ &+ \left[1 - \left(\frac{r_+}{r} \right)^{d-3} \right]^{-1} \left[1 - \left(\frac{r_-}{r} \right)^{d-3} \right]^{b-1} dr^2 \\ &+ r^2 \left[1 - \left(\frac{r_-}{r} \right)^{d-3} \right]^b d\Omega_{d-2}^2, \\ e^{a\phi} &= \left[1 - \left(\frac{r_-}{r} \right)^{d-3} \right]^{-(d-3)b/2}, \\ F_{d-2} &= Q \varepsilon_{d-2}, \end{aligned} \quad (4)$$

where ε_{d-2} is the volume form on the unit $(d-2)$ -sphere, the constant b is

$$b = 2p/(d-2)(p+1), \quad (5)$$

and the charge Q is related to r_{\pm} by

$$Q^2 = \frac{(d-3)(d+p-2)}{2(p+1)} (r_+ r_-)^{d-3}. \quad (6)$$

Thus, one has the non-dilatonic black p -brane solutions in the action (1) [8]:

$$ds_{d+p}^2 = e^{2m\phi} dy^i dy^i + e^{2n\phi} ds_d^2, \quad (7)$$

where $i = 1, 2, \dots, p$, and

$$m = -\frac{\sqrt{2(d-2)}}{\sqrt{p(d+p-2)}}, \quad n = -\frac{mp}{d-2}. \quad (8)$$

From Eqs. (4)-(8), the Hawking temperature and the Bekenstein-Hawking entropy per unit volume of p -branes for the black p -branes (7) are easily obtained:

$$T = \frac{d-3}{4\pi r_+} \left[1 - \left(\frac{r_-}{r_+} \right)^{d-3} \right]^{1/(p+1)}, \quad (9)$$

$$S = \frac{\Omega_{d-2}}{4} r_+^{d-2} \left[1 - \left(\frac{r_-}{r_+} \right)^{d-3} \right]^{p/(p+1)}, \quad (10)$$

where Ω_{d-2} is the volume of the unit $(d-2)$ -sphere. The ADM mass per unit volume of p -branes is found to be

$$M = \frac{\Omega_{d-2}}{16\pi} \left[(d-2)r_+^{d-3} + \frac{d-2-p(d-4)}{p+1} r_-^{d-3} \right], \quad (11)$$

which satisfies the first law of thermodynamics,

$$dM = TdS + \Phi dQ, \quad (12)$$

where $\Phi = \Omega_{d-2}Q/[4\pi(d-3)r_+^{d-3}]$ is the chemical potential corresponding to the conservative charge Q . According to the formula $C_Q \equiv (\partial M/\partial T)_Q$, the heat capacity per unit volume of p -branes is

$$C_Q = -\frac{\Omega_{d-2}r_+}{4} \frac{\left[1 - \left(\frac{r_-}{r_+} \right)^{d-3} \right]^{p/(p+1)}}{\left[1 - \frac{p+2d-5}{p+1} \left(\frac{r_-}{r_+} \right)^{d-3} \right]} \times \left[(d-2)r_+^{d-3} - \frac{d-2-p(d-4)}{p+1} r_-^{d-3} \right]. \quad (13)$$

When the extremal limit ($r_- = r_+$) is approached, the temperature, entropy, and the heat capacity approach zero. When $1 - (p+2d-5)(r_-/r_+)^{d-3}/(p+1) = 0$, the heat capacity diverges, which corresponds to the critical point of Davies in Kerr-Newman black holes [10].

In a self-gravitating thermodynamic system, in general, the thermodynamic ensembles are not equivalent [11]. Hence, the critical behavior and stability of the system are different in the different environments (implying the different ensembles). In order to discuss the critical

behavior of an isolated black p -brane, it is reasonable to choose the microcanonical ensemble [12-14]. In this ensemble, the proper Massieu function, which can describe completely the equilibrium state of a thermodynamic system, is the entropy of the system. For the black p -branes (7), rewriting Eq. (12), one has

$$dS = \beta dM - \varphi dQ, \quad (14)$$

where $\beta = T^{-1}$ and $\varphi = \beta\Phi$. Applying the fluctuation theory of equilibrium thermodynamic under the specified environments [11-14] to the black p -branes (7), it follows from Eq. (14) that the intrinsic variables $x_i = \{M, Q\}$ and the conjugate variables $X_i = \{\beta, -\varphi\}$ in the microcanonical ensemble. Thus the eigenvalues corresponding to the fluctuation modes β and φ are

$$\lambda_m = \left(\frac{\partial M}{\partial \beta} \right)_Q = -T^2 C_Q, \quad (15)$$

$$\lambda_q = - \left(\frac{\partial Q}{\partial \varphi} \right)_M = -TK_M, \quad (16)$$

respectively, where

$$\begin{aligned} K_M &\equiv \beta \left(\frac{\partial Q}{\partial \varphi} \right)_M \\ &= \frac{4\pi(d-3)r_+^{d-3}}{\Omega_{d-2}} \left[1 - \frac{d-2-p(d-4)}{(p+1)(d-2)} \left(\frac{r_-}{r_+} \right)^{d-3} \right] \\ &\quad \left\{ \left[1 + \frac{d-2-p(d-4)}{(p+1)(d-2)} \left(\frac{r_-}{r_+} \right)^{d-3} \right] \right. \\ &\quad + \frac{2}{(d-3)} \left(\frac{r_-}{r_+} \right)^{d-3} \left[\frac{d-3}{p+1} - \frac{d-2-p(d-4)}{(p+1)(d-2)} \right. \\ &\quad \times \left. \left. \left(1 - \frac{p+d-2}{p+1} \left(\frac{r_-}{r_+} \right)^{d-3} \right) \right] \right. \\ &\quad \times \left. \left. \left[1 - \left(\frac{r_-}{r_+} \right)^{d-3} \right]^{-1} \right\}^{-1}. \end{aligned} \quad (17)$$

By using the fluctuation theory, we obtain the nonvanishing second moments of fluctuations,

$$\langle \delta\beta\delta\beta \rangle = -k_B \frac{\beta^2}{C_Q}, \quad \langle \delta\varphi\delta\varphi \rangle = -k_B \frac{\beta}{K_M},$$

$$\langle \delta\beta\delta\Phi \rangle = k_B \frac{\beta\Phi}{C_Q}, \quad \langle \delta\Phi\delta\Phi \rangle = -k_B \left(\frac{T}{K_M} + \frac{\Phi^2}{C_Q} \right). \quad (18)$$

Obviously, the two eigenvalues approach zero and all of these second moments diverge when the extremal limit ($r_- = r_+$) is approached. As in the ordinary thermodynamics, the divergence of second moments means that the extremal limit is a critical point and a second-order phase transition takes place from the extremal to nonextremal black p -branes. As is well known, the extremal black p -braes are very different from the nonextremal in many aspects, such as the thermodynamic description [15] and geometric structures [16]. In particular, it has been shown that the extremal black p -branes are supersymmetric and the supersymmetry is absent for the nonextremal black p -branes [8,9]. So the occurrence of phase transition are consistent with the changes of symmetry. The extremal and nonextremal black p -branes are two different phases. The extremal black p -branes are in the disordered phase and the nonextremal black p -branes in the ordered phase. The order parameters of the phase transition can be defined as the differences of the conjugate variables between the two phases [12,13], such as $\eta_\beta = \beta_+ - \beta_-$ and $\eta_\varphi = \varphi_+ - \varphi_-$ can be regarded as the order parameters of black p -branes, where the suffixes “+” and “-” mean that the quantity is taken at the r_+ and r_- , respectively. The second-order derivatives of entropy with respect to the intrinsic variables are the inverse eigenvalues,

$$\zeta_m \equiv \left(\frac{\partial^2 S}{\partial M^2} \right)_Q = \lambda_m^{-1} = -\frac{\beta^2}{C_Q}, \quad (19)$$

$$\zeta_q \equiv \left(\frac{\partial^2 S}{\partial Q^2} \right)_M = \lambda_q^{-1} = -\frac{\beta}{K_M}. \quad (20)$$

Correspondingly, we can define the critical exponents of these quantities as follows [17],

$$\begin{aligned} \zeta_m &\sim \varepsilon_M^{-\alpha} \quad (\text{for } Q \text{ fixed}), \\ &\sim \varepsilon_Q^{-\psi} \quad (\text{for } M \text{ fixed}), \end{aligned} \quad (21)$$

$$\begin{aligned} \zeta_q &\sim \varepsilon_M^{-\gamma} \quad (\text{for } Q \text{ fixed}), \\ &\sim \varepsilon_Q^{-\sigma} \quad (\text{for } M \text{ fixed}), \end{aligned} \quad (22)$$

$$\eta_\varphi \sim \varepsilon_M^\beta \quad (\text{for } Q \text{ fixed}),$$

$$\sim \varepsilon_Q^{\delta^{-1}} \quad (\text{for } M \text{ fixed}), \quad (23)$$

where ε_M and ε_Q represent the infinitesimal deviations of M and Q from their limit values. These critical exponents are found to be

$$\alpha = \psi = \gamma = \sigma = \frac{p+2}{p+1}, \quad \beta = \delta^{-1} = -\frac{1}{p+1}. \quad (24)$$

The critical exponents β and δ^{-1} are negative, which shows the fact that the order parameter η_φ diverges at the extremal limit. This is because the critical temperature is zero in this phase transition. It is easy to check that these critical exponents satisfy the scaling laws of the “first kind,”

$$\alpha + 2\beta + \gamma = 2, \quad \beta(\delta - 1) = \gamma, \quad \psi(\beta + \gamma) = \alpha. \quad (25)$$

That scaling laws (25) hold for the black p -branes is related to the fact that the black p -brane entropy (10) is a homogeneous function, satisfying

$$S(\lambda M, \lambda Q) = \lambda^{(d-2)/(d-3)} S(M, Q), \quad (26)$$

where λ is a positive constant. On the other hand, in an ordinary thermodynamic system, an important physical quantity related to phase transitions is the two-point correlation function, which has generally the form for a large distance [17],

$$G(r) \sim \frac{\exp(-r/\xi)}{r^{\bar{d}-2+\eta}}, \quad (27)$$

where η is the Fisher’s exponent, \bar{d} is the effective spatial dimension of the system under consideration, and ξ is the correlation length and diverges at the critical point. Similarly, the critical exponents of the correlation length for black p -branes can be defined as:

$$\begin{aligned} \xi &\sim \varepsilon_M^{-\nu} \quad (\text{for } Q \text{ fixed}), \\ &\sim \varepsilon_Q^{-\mu} \quad (\text{for } M \text{ fixed}). \end{aligned} \quad (28)$$

Combining with those in Eq. (25), these critical exponents form the scaling laws of the “second kind”

$$\nu(2 - \eta) = \gamma, \quad \nu\bar{d} = 2 - \alpha, \quad \mu(\beta + \gamma) = \nu. \quad (29)$$

Because of the absence of quantum theory of gravity, we have not yet the correlation function of quantum black holes. Here we use the correlation function of scalar fields on the background of these black p -branes to mimic the one of black p -branes (from the obtained result below, it seems an appropriate approach to study the critical behavior of black holes at the present time). From the work of Traschen [18] who studied the behavior of scalar fields on the background of Reissner-Nordström black holes, it is found that the inverse surface gravity of the black hole plays the role of the correlation length of scalar fields. For the black p -branes, this conclusion holds as well. With the help of the surface gravity of black p -branes (7), we obtain

$$\nu = \mu = \frac{1}{p+1}. \quad (30)$$

Substituting (30) into (29), we find

$$\eta = -p, \quad \bar{d} = p. \quad (31)$$

When $p = 0$, the black p -branes (7) reduce to the non-dilatonic d -dimensional black holes. In this case, these critical exponents become

$$\begin{aligned} \alpha = \psi = \gamma = \sigma = 3/2, \quad \beta = \delta^{-1} = -1/2, \\ \eta = -1, \quad \bar{d} = 1, \end{aligned} \quad (32)$$

independent of the dimensionality of spacetime. These critical exponents are exactly the same as those of three dimensional Bañados-Teitelboim-Zanelli (BTZ) black holes [13]. Recall the fact that the BTZ black holes are also exact non-dilatonic black hole solutions in string theory [19], we find that these critical exponents are universal for non-dilatonic black holes, an important feature of critical behavior in the non-dilatonic black holes. For the dilatonic black holes with the coupling constant a obeying (3), we find that the effective spatial dimension is also p (it is one for $p=0$). For a general a , the scaling laws still hold,

but these critical exponents and effective dimension will depend on the coupling constant a and the dimension d of spacetime. In this case, the effective spatial dimension is

$$\bar{d} = \frac{(d-2)^2 a^2}{2(d-3)^2 - (d-2)a^2}. \quad (33)$$

This statement is also valid for the dilatonic black p -branes [14].

Summarizing the above, we have the following conclusions: (1) The extremal limit of dilatonic and non-dilatonic black p -branes is critical point and corresponding critical exponents obey the scaling laws. (2) For the non-dilatonic black holes and black strings, the effective spatial dimension is one. This result is also reached in the BTZ black holes and 3-dimensional black strings [13]. (3) For the non-dilatonic black p -branes (black string for $p = 1$), the effective dimension is p , so does it for the dilatonic black holes produced by the double-dimensional reduction of the non-dilatonic black p -branes. (4) For other dilatonic black holes and black p -branes, the effective spatial dimension depends on the parameters in theories. Furthermore, near the extremal limit of the non-dilatonic black p -branes, from Eqs. (9)-(11), we have

$$S \sim T^p, \quad M - M_{\text{ext}} \sim T^{p+1}, \quad (34)$$

where M_{ext} is the ADM mass of extremal black p -branes. Notice that the ADM mass and entropy of black p -branes are extensive quantities with respect to the volume of p -branes. Thus, near the extremal limit, the thermodynamic properties of non-dilatonic black p -branes can be described by the blackbody radiation in $(1+p)$ dimensions, which also further verify that the effective spatial dimension is p . For the dilatonic black holes (4), equation (34) is also valid. Although the entropy of dilatonic black holes is not an extensive quantity, the entropy can be regarded as the entropy density of the non-dilatonic black p -branes. Thus it seems to imply that these dilatonic black hole entropy can also be explained as the way of p -branes, although the string coupling becomes very large in this case.

Recall the recent progress in understanding entropy of black holes [5-7], in which the constant dilaton field seems to be a necessary condition. Therefore, our conclusions are

in complete agreement with the result of these investigations. Further we also give an interpretation why the Bekenstein-Hawking entropy may be given a simple world volume interpretation only for the non-dilatonic p -branes (including the non-dilatonic black holes). In addition, more recently, Horowitz and Polchinski [20] have proposed a correspondence principle, which states that (i) when the size of the black hole horizon drops below the size of a string, the typical black hole state becomes a typical state of strings and D-branes with the same charges, and (ii) the mass does not change abruptly during the transition. This principle connecting black holes to weakly coupled strings and D-branes provides a statistical interpretation of entropy of black holes (including dilatonic black holes). Therefore, our phase transition in fact corresponds to the transition between black hole description and string description.

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